

Dynamical Symmetry Breaking in a Gauge Theory “Thirring Model” *

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Abstract

Dynamical fermion mass generation is studied in the three-dimensional Thirring model reformulated as a gauge theory by introducing hidden local symmetry. The analysis by use of Schwinger-Dyson equation is shown to exhibit a critical behavior as the number N of four-component fermions approaches $N_{\text{cr}} = 128/3\pi^2$.

When we consider the models of massless self-interacting fermions, a central issue is the dynamical fermion mass generation. If it is indeed the case, intriguing detailed questions are whether the model of interest is shown to exhibit a critical behavior as the number N of fermions approaches N_{cr} and, in three dimensions, whether the pattern of dynamically generated masses preserves the parity or not.

In this report we address ourselves to the problem of dynamical symmetry breaking in the Thirring model of N massless four-component fermions. (For the notations and the detailed calculations, see Ref.[1].)

$$\mathcal{L}_{\text{Thi}} = \sum_a \bar{\psi}_a i\gamma^\mu \partial_\mu \psi_a - \frac{G}{2N} \sum_{a,b} \bar{\psi}_a \gamma^\mu \psi_a \bar{\psi}_b \gamma_\mu \psi_b, \quad (1)$$

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where ψ_a is a four-component Dirac fermion and the indices a, b run over N fermion species. Since the form of four-fermion interaction term is a contact term between vector currents, a well-known technique to facilitate $1/N$ -expansion is to rewrite Eq.(1) by introducing an auxiliary vector field A^μ such as [2]~[6]

$$\mathcal{L}_{\text{aux}} = \sum_a \bar{\psi}_a i\gamma^\mu (\partial_\mu - \frac{i}{\sqrt{N}} A_\mu) \psi_a + \frac{1}{2G} A_\mu A^\mu. \quad (2)$$

Gauge-noninvariant as the above Lagrangian is, however one may be tempted to regard A_μ as a gauge field. A systematic way to construct the $U(1)$ gauge theory, but is gauge equivalent to Eq.(2) is to elicit the fictitious Goldstone degree (or equivalently the Stückelberg field) based on the principle of hidden local symmetry [9], [7, 1, 8]:

$$\mathcal{L}_{\text{HLS}} = \sum_a \bar{\psi}_a i\gamma^\mu D_\mu \psi_a + \frac{1}{2G} (A_\mu - \sqrt{N} \partial_\mu \phi)^2. \quad (3)$$

It is obvious that Eq.(3) possesses a $U(1)$ gauge symmetry and the gauge-fixed (unitary gauge) form of it exactly coincides with Eq.(2), so does the original Thirring model in Eq.(1).

Now that we find a gauge-invariant formulation of the Thirring model, we have the privilege to choose the gauge appropriate for our particular purpose. Here, instead of the unitary gauge notorious for loop calculations, let us consider the nonlocal R_ξ gauge at the Lagrangian level

$$\mathcal{L}_{\text{GF}} = -\frac{1}{2} \left(\partial_\mu A^\mu + \sqrt{N} \frac{\xi(\partial^2)}{G} \phi \right) \frac{1}{\xi(\partial^2)} \left(\partial_\nu A^\nu + \sqrt{N} \frac{\xi(\partial^2)}{G} \phi \right), \quad (4)$$

where the gauge fixing parameter ξ has the momentum- (derivative-) dependence. It is straightforward to prove the possession of the BRS symmetry in spite of the nonlocality of ξ and thereby it guarantees the S-matrix unitarity. Another intriguing point is that, in the combined Lagrangian of Eqs. (3) and (4), the fictitious Nambu-Goldstone boson ϕ is completely decoupled independently of the specific form of $\xi(\partial^2)$.

In (2+1) dimensions, our Lagrangian in Eq.(3) is invariant under the parity

$$\psi_a(x) \mapsto \psi'_a(x') = i\gamma^3 \gamma^1 \psi_a(x), \quad A_\mu(x) \mapsto A'_\mu(x') = (-1)^{\delta_{\mu 1}} A_\mu(x). \quad (5)$$

and the so-called global “chiral” transformation

$$\psi_a \mapsto \psi'_a = \left(\exp(i\omega^{i\alpha} \frac{\Sigma^i}{2} \otimes T^\alpha) \psi \right)_a, \quad (6)$$

where $\Sigma^0 = I$, $\Sigma^1 = -i\gamma^3$, $\Sigma^2 = \gamma^5$, $\Sigma^3 = -\gamma^5\gamma^3$ and T^α denote the generators of $U(N)$. The question we shall address from now on is “which symmetry is broken dynamically?” In concern with the parity, first issue is whether one can take the regularization to keep both the $U(1)$ gauge symmetry and the parity or not. Since our gauge action in tree level has the parity-conserving mass and the number of two component Dirac fermion species is even, the parity need not be violated by appropriate regulator. For example, the introduction of parity-conserving Pauli-Villars regulator leads to the parity-invariant effective action for the gauge field as have done in (2+1)D quantum electrodynamics(QED₃) [10]. Another question is whether the pattern of dynamically-generated fermion mass involves the parity violating mass ($-m\bar{\psi}_a\gamma^5\gamma^3\psi_a$) or not. Though at this stage we do not yet know whether the dynamical symmetry breaking really occurs or not, in this gauge-invariant formulation of the Thirring model, such symmetry breaking pattern is proven to be energetically unfavorable by using the exact argument in Ref.[11]. Namely, since the tree-level gauge action corresponding to Eq.(3) is real and positive semi-definite in Euclidean space, energetically favorable is a parity conserving configuration consisting of half the 2-component fermions acquiring equal positive masses and the other half equal negative masses.

According to the above arguments, the pattern of symmetry breaking we shall consider is not the parity but the chiral symmetry, i.e., of which the breaking is $U(2N) \rightarrow U(N) \times U(N)$. Thus we investigate the dynamical mass of the type $m\bar{\psi}\psi$ in the Schwinger-Dyson equation, giving

$$(A(-p^2)-1)\not{p}-B(-p^2) = -\frac{1}{N} \int \frac{d^D q}{i(2\pi)^D} \gamma_\mu \frac{A(-q^2)\not{q} + B(-q^2)}{A^2(-q^2)q^2 - B^2(-q^2)} \Gamma_\nu(p, q) iD^{\mu\nu}(p-q), \quad (7)$$

where the full fermion propagator is written as $S(p) = i[A(-p^2)\not{p} - B(-p^2)]^{-1}$, and $\Gamma_\nu(p, q)$ and $D_{\mu\nu}(p - q)$ denote the full vertex function and the full gauge boson propagator, respectively. Task is, by employing some appropriate approximations, to reduce Eq.(7) to the tractable integral equation for the mass function $M(-p^2) = B(-p^2)/A(-p^2)$. First, we here adopt the $1/N$ expansion for $\Gamma_\nu(p, q)$ and $D_{\mu\nu}(p - q)$ under a nonlocal R_ξ gauge, i.e., they are the bare vertex and the one-loop vacuum polarization of massless fermion loop at the $1/N$ leading order. Then the Schwinger-Dyson equation (7) becomes the coupled integral equations for $A(-p^2)$ and $B(-p^2)$. They support a trivial solution $A(-p^2) = 1$ and $B(-p^2) = 0$ at the $1/N$ leading order,

however, as was realized in QED₃, we expect to find a nonperturbative nontrivial solution by examining them for finite N .

A way is, by use of the freedom of gauge choice, to require $A(-p^2) = 1$ in a Schwinger-Dyson equation for $A(-p^2)$. Then this gauge fulfills the consistency between the bare vertex approximation and the Ward-Takahashi identity for the hidden local $U(1)$ symmetry (or the current conservation), i.e., $A(0) = 1$. The specific form of the gauge is determined by a Schwinger-Dyson integral equation for $A(-p^2)$, and it reduces the coupled Schwinger-Dyson equations into a single equation for $B(-p^2)$ which turns out to be a mass function, i.e., $M(-p^2) = B(-p^2)$:

$$B(p^2) = \frac{1}{N} \int_0^{\Lambda^{D-2}} d(q^{D-2}) K(p, q; G) \frac{q^2 B(q^2)}{q^2 + B^2(q^2)}, \quad (8)$$

where Λ is ultraviolet cutoff and the kernel $K(p, q; G)$ is given by

$$K(p, q; G) = \frac{1}{(D-2)2^{D-1}\pi^{(D+1)/2}\Gamma(\frac{D-1}{2})} \int_0^\pi d\theta \sin^{D-2} \theta d(k^2) [D - \eta(k^2)], \quad (9)$$

with $k^2 = p^2 + q^2 - 2pq \cos \theta$. The gauge fixing parameter ξ is a function of k^2 such as

$$\eta(k^2) = \frac{-\xi(k^2)C_D^{-1}k^{D-2} + k^2}{\xi(k^2)G^{-1} + k^2} \quad (10)$$

$$= (D-2) \left[\left(1 + \frac{Gk^{D-2}}{C_D} \right) {}_2F_1 \left(1, 1 + \frac{D}{D-2}, 2 + \frac{D}{D-2}; -\frac{Gk^{D-2}}{C_D} \right) - 1 \right], \quad (11)$$

where $C_D^{-1} \equiv \frac{2 \text{tr} I}{(4\pi)^{D/2}} \Gamma \left(2 - \frac{D}{2} \right) B \left(\frac{D}{2}, \frac{D}{2} \right)$ and ${}_2F_1(a, b, c; z)$ the hypergeometric function. Note that the kernel $K(p, q; G)$ is positive-definite for positive arguments p, q, G and is symmetric under the exchange of p and q .

Now let us show the existence of the nontrivial solution for the Schwinger-Dyson equation (8) when $2 < D < 4$. Since we have in mind the continuous phase transition, the solution of our interest is the nontrivial solution which starts to exist without gap in the vicinity of the phase transition point. Such a bifurcation point is identified by the existence of an infinitesimal solution $\delta B(p^2)$ around the trivial solution $B(p^2) = 0$. In terms of dimensionless variables ($p = \Lambda x^{1/(D-2)}$, $\delta B(p^2) = \Lambda \Sigma(x)$, $g = G/\Lambda^{2-D}$), Eq.(8) is reduced to a linearized integral equation

$$\Sigma(x) = \frac{1}{N} \int_{\sigma_m}^1 dy K(x^{1/(D-2)}, y^{1/(D-2)}; g) \Sigma(y), \quad (12)$$

where $\sigma_m = (m/\Lambda)^{D-2}$ ($0 < \sigma_m \leq 1$) is the rescaled infrared cutoff and in fact m is nothing but the dynamically generated mass by the normalization $m = \delta B(m^2)$. We can rigorously prove that, *if N is equal to the maximal eigenvalue of the kernel (9), then there exists a nontrivial solution $\Sigma(x)$* . Hence, for a given σ_m , each line $N(g, \sigma_m)$ on (N, g) plane depicts a line of equal dynamically-generated mass $m = \Lambda \sigma_m^{1/(D-2)}$. Therefore, the critical line is defined by $N_{\text{cr}}(g) = \lim_{\sigma_m \rightarrow 0} N(g, \sigma_m)$ which separates the broken phase from the symmetric phase. It is difficult to obtain the explicit form of the critical line $N_{\text{cr}}(g)$ for arbitrary g , however we can get it in the limit of infinite four-fermion coupling constant, $g \rightarrow \infty$. In this limit, the bifurcation equation (12) in (2+1)D is rewritten into a differential equation

$$\frac{d}{dx} \left(x^2 \frac{d\Sigma(x)}{dx} \right) = -\frac{32}{3\pi^2 N} \Sigma(x), \quad (13)$$

plus the infrared boundary condition $\Sigma'(\sigma_m) = 0$ and the ultraviolet one $[x\Sigma'(x) + \Sigma(x)]_{x=1} = 0$. When $N > N_{\text{cr}} \equiv 128/3\pi^2$, there is no nontrivial solution of Eq.(13) satisfying the boundary conditions, while for $N < N_{\text{cr}}$ the following bifurcation solutions exist:

$$\Sigma(x) = \frac{\sigma_m}{\sin(\frac{\omega}{2}\delta)} \left(\frac{x}{\sigma_m} \right)^{-\frac{1}{2}} \sin \left\{ \frac{\omega}{2} \left[\ln \frac{x}{\sigma_m} + \delta \right] \right\}, \quad (14)$$

where $\omega \equiv \sqrt{N_{\text{cr}}/N - 1}$, $\delta \equiv 2\omega^{-1} \arctan \omega$ and σ_m is given by the ultraviolet boundary condition:

$$\frac{\omega}{2} \left(\ln \frac{1}{\sigma_m} + 2\delta \right) = n\pi, \quad n = 1, 2, \dots \quad (15)$$

The solution with $n = 1$ is the nodeless (ground state) solution whose scaling behavior is read from Eq.(15):

$$\frac{m}{\Lambda} = e^{2\delta} \exp \left[-\frac{2\pi}{\sqrt{N_{\text{cr}}/N - 1}} \right]. \quad (16)$$

The critical four-fermion number $N_{\text{cr}} = 128/3\pi^2$ is the same as the one in QED₃ with the nonlocal gauge.

It is turn to comment briefly on the dynamically generated mass of the gauge boson and the dual transformation. The vector (gauge) boson is merely an auxiliary field at the tree level, however it turns out to be propagating by obtaining the kinetic term through fermion loop effect when the fermion acquires the dynamical mass. In (2+1) dimensions the pole mass M_V of the dynamical gauge boson is given by

$$\frac{1}{2\pi} \left[\frac{4m^2 + M_V^2}{2M_V} \tan^{-1} \frac{M_V}{2m} - m \right] = G^{-1}, \quad (17)$$

which always satisfies a condition that $M_V < 2m$. Once the Thirring model is understood as a gauge theory (3), its effective theory can be expressed in terms of dual antisymmetric-tensor field $H_{\mu_1 \dots \mu_{D-2}}$ of rank $D - 2$, which is actually a composite of fermions $\epsilon^{\mu_1 \dots \mu_D} \partial_{\mu_2} H_{\mu_3 \dots \mu_D} = -\frac{\sqrt{G}}{(D-1)\sqrt{N}} \sum_a \bar{\psi}_a \gamma^{\mu_1} \psi_a$. This relation implies the current conservation at the quantum level, which is a crucial property for avoiding peculiarity in the lattice study of the Thirring model [13]. In (1+1) dimensions the above relation is nothing but the one of the bosonization, however extension to higher-dimensional case by integrating out the fermions leads to nonlocal bosonic Lagrangian [12]. In (2+1) dimensions the dual gauge field H_μ shares exactly the same pole structure with the gauge field A_μ irrespectively of the phase.

Analytic studies in the scheme of the Schwinger-Dyson equations with an $1/N$ expansion yield the existence of the phase transition line on the plane of couplings (N, g) and predict the critical number of the four component fermion being $N_{\text{cr}} = 128/3\pi^2$ in the $g \rightarrow \infty$ limit. In order to check these intriguing questions, an appropriate systematic method is lattice simulation. Here we present the status of the simulation of lattice gauge theory [14]. The continuum Lagrangian is a $U(1)$ gauge theory, and then the discretized version of Eq.(3) leads to a lattice gauge theory expressed in terms of a link variable $e^{i\theta_\mu(x)}$ and staggered fermions:

$$\mathcal{L}_L = \sum_a \Phi_a^\dagger (M^\dagger M)^{-1} \Phi_a - N\beta \sum_\mu \cos(\phi(x+\mu) - \phi(x) + \theta_\mu(x)), \quad (18)$$

where $\beta = 1/g$ and

$$M_{x,y} = \frac{1}{2} \sum_\mu \eta_\mu(x) (e^{i\theta_\mu(x)} \delta_{y,x+\mu} - e^{-i\theta_\mu(x-\mu)} \delta_{y,x-\mu}) + m \delta_{x,y}. \quad (19)$$

Note that one can not discriminate whether this lattice Lagrangian (18) at the tree level lies in the compact formulation or in the noncompact one since there is no gauge kinetic term at the tree level. This issue is rather subtle because each continuum limit may arrives at different continuum theory as has been done in QED₃.

So far, simulations for $N = 2, 4$ and 6 systems on a 8^3 lattice volume with bare fermion mass $m = 0.05, 0.025$ and 0.0125 are completed by use of hybrid Monte Carlo algorithm and preliminary data on a 16^3 lattice for $N = 2$ and 6 are prepared for finite volume effect study [14]. Numerical simulation results on a 8^3 lattice are summarized in Fig. 1 where the plot points are those obtained by $m \rightarrow 0$ extrapolation. $N = 2$ shows smooth change of the chiral condensate and a long tail into weak coupling

regime, $N = 4$ shows a complex behavior of the condensate near the critical point, and $N = 6$ shows an abrupt change of the condensate which can be interpreted as a mean field type phase transition. The result from a coarse-grained lattice alludes to the existence of the critical fermion number N_{cr} ($2 < N_{\text{cr}} < 6$) for which the chiral behavior changes its character. It is consistent with the result in the continuum Thirring model reformulated as a gauge theory. Preliminary as the result is, the above suggests possible existence of the critical N_{cr} in the lattice gauge formulation of the Thirring model. Data on a 16^3 lattice tells us that, though the dependence of the chiral condensate on both the size of the lattices and the extrapolation method is considerable quantitatively (particularly in the very weak coupling region of $N = 2$ case), the overall qualitative nature of the chiral condensate does not seem to be changed. It is interesting to wait further lattice result of $N = 6$ case based on the auxiliary vector field [13] for the comparison with the above [14].

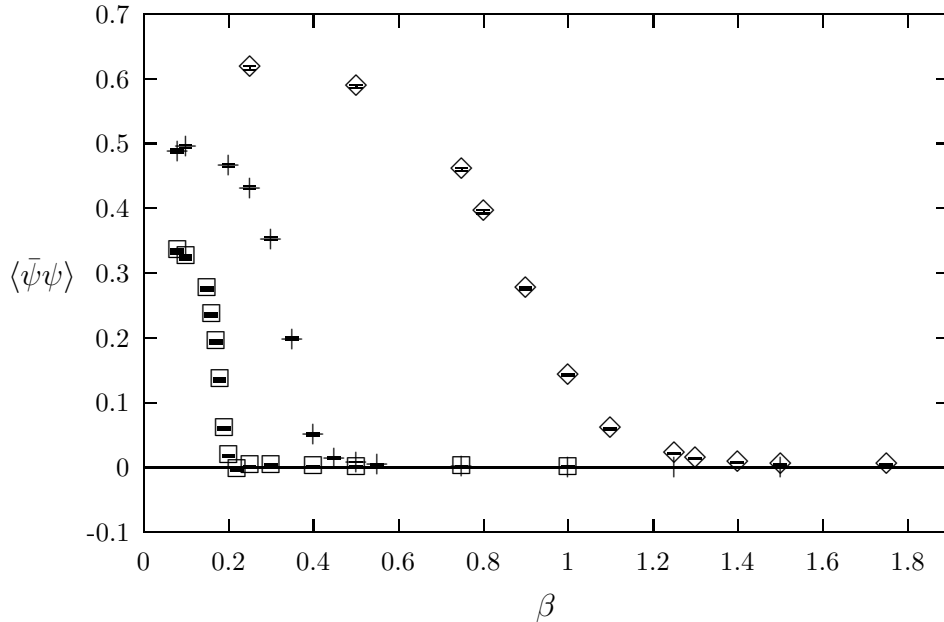


Figure 1: The chiral condensate, $\langle \bar{\psi}\psi \rangle$, vs. β for $N = 2$ (\diamond points), $N = 4$ (+ points), and $N = 6$ (\square points).

In this report, we have reviewed how to reformulate the Thirring model as a gauge theory both on continuum and lattice, and, in such context, the dynamical symmetry breaking for N four component Dirac fermions in (2+1)D. The Schwinger-Dyson approach in cooperation with $1/N$ expansion reaches a rigorous proof of the existence

of the chiral phase transition at a certain number of N and g under the light shed by the hidden local symmetry. The application of Vafa-Witten theorem forces the parity preservation in the dynamically-generated fermion mass spectra and the argument of parity-violating anomaly prevents the parity violation in gauge sector. In the infinite coupling limit ($g \rightarrow \infty$), the Schwinger-Dyson equation was solved analytically and yields the critical fermion number $N_{\text{cr}} = 128/3\pi^2$ in perfect agreement with N_{cr} in QED₃. More systematic analysis of the Schwinger-Dyson equations is in progress [15]. According to the present lattice simulations on 8^3 and 16^3 lattices, the lattice gauge formulation seems to support the existence of N_{cr} ($2 < N_{\text{cr}} < 6$), which should be tested again by further lattice simulation [16].

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